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Initial and Final Value Theorem for Laplace-Weierstrass Transform

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Abstract

Integral transform method have proved to be the great importance in solving boundary value problems of mathematical physics and partial differential equation.We had defined classical Laplace-Weierstrass transform in generalized sense. In this paper we have proved initial and final value theorem for Laplace-Weierstrass transform. The results are used to solve boundary value problems of partial differential equation and in mathematical physics.

Keywords:

Laplace transform; Weierstrass transform; Laplace-Weierstrass transform; Initial value theorem; Final value theorem

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1. Introduction

In mathematical analysis, the Initial value theorem is a theorem used to relate frequency domain expression to the time domain behavior as time approaches to zero. And the Final value theorem is one of several similar theorems used to relate frequency domain expression to the time domain behavior as time approaches infinity. The Initial and Final value theorems are obtained as the complex variable of the transform approaches 0 or ∞ in absolute value inside a wedge region in the right half plane. By an 'Initial (Final) value theorem, we mean a theorem that relates the Initial (Final) value of a distribution to the Final (Initial) value of the transform. The Laplace transform is denoted and defined by

$$L[f(t)] = \langle f(t), e^{-st} \rangle \qquad (1.1)$$

The Weierstrass transform is denoted and defined by

$$W[f(\Im)] = \frac{1}{\sqrt{4\pi}} \left\langle f(\Im), e^{-\frac{\Im^2}{4}} \right\rangle \qquad (1.2)$$

The kernel $K(s-\Im,1)$ as a function of $\Im$ is a member of $W(w,z)$ if and only if $w<\operatorname{Re}s<z$, also if t is fixed with $0<t<1$, then $K(s-\Im,t)$ as a function of $\Im$ is a member of $W(w,z)$ for every fixed $s\in\wp'$ and every w and z. Furthermore, conventional differentiation is a continuous linear mapping of $W(w,z)$ into itself; generalized differentiation is a continuous linear mapping of $W'(w,z)$ into itself. Thus, the Laplace-Weierstrass transform is denoted and defined as

$$LW[f(t,\Im)]=\frac{1}{\sqrt{4\pi}}\left\langle f(t,\Im),e^{-st-\frac{\Im^2}{4}}\right\rangle \qquad (1.3)$$

In this paper, we shall establish Initial and Final value theorems for the generalized Laplace-Weierstrass transform. In section 2, we have given Initial value theorem for the generalized Laplace-Weierstrass transform .Final value theorem for generalized Laplace-Weierstrass transform is proved in section 3. The setting for these Initial and Final value theorems is motivated by the Initial and Final value results of Doetsch [2]. Initial and Final value theorems for various transform are introduced in [3, 4, 5].
Our notations and terminology are the same as used in Zemanian [7].

2. Initial value theorem for Laplace-Weierstrass transform

Here we take the function $f(t,\Im)$ that is absolutely integrable over $0<t<\infty$ and $0<\Im<\infty$. We shall also impose that $f(t,\Im)$ is a right sided and locally integrable function, which satisfisfies the following conditions:-

i) $f(t,\Im)$=0, for $0<t<T$, $0<\Im<\tau$

ii) There exists a real number s such that $f(t,\Im)e^{-st-\frac{\Im^2}{4}}$ is absolutely integrable over $0<t<\infty$ and $0<\Im<\infty$.

2.1 Theorem:-

For a locally integrable function $f(t,\Im)$ satisfying above conditions with $T=0$ and existence of any complex constant B and a real numbers m and n such that,

$i)\ m>-1$

$ii)\ n>-1$

$$iii)\ \lim_{\substack{t\to 0^+\\ \Im\to 0^+}}\frac{\Gamma m+1\ \ \Gamma n+\frac{1}{2}\,(2)^n}{t^m\,\Im^n}f(t,\Im)=B$$

then, $\lim_{s\to\infty}s^{m+1}LW\{f(t,\Im)\}=B$

Proof: We know by Zemanian [7] pp. 243 and Lokenath-Debnath [1] pp.136,

i) For $m>-1$ and $s>0$

$$\int_0^\infty t^m\,e^{-st}\,dt=\frac{\Gamma m+1}{s^{m+1}}$$

ii) For $n>-1$

$$\int_0^\infty \Im^n e^{-\frac{\Im^2}{4}} d\Im = (2)^n \Gamma \frac{n+1}{2}$$

Hence,

$$s^{m+1} LW\{f(t,\Im)\} - B = s^{m+1} \int_0^\infty \int_0^\infty f(t,\Im) e^{-st-\frac{\Im^2}{4}} dt\, d\Im - \frac{B s^{m+1}}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \int_0^\infty \int_0^\infty t^m \Im^n e^{-st-\frac{\Im^2}{4}} dt\, d\Im$$

$$= s^{m+1} \int_0^\infty \int_0^\infty \left[f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right] e^{-st-\frac{\Im^2}{4}} dt\, d\Im$$

$$\left| s^{m+1} LW\{f(t,\Im)\} - B \right| \le s^{m+1} \{ \int_0^T \int_0^\tau \left| f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}} dt\, d\Im$$

$$+ \int_T^\infty \int_0^\tau \left| f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}} dt\, d\Im$$

$$+ \int_0^T \int_\tau^\infty \left| f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}} dt\, d\Im$$

$$+ \int_T^\infty \int_\tau^\infty \left| f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}} dt\, d\Im \}$$

By Widder [6] pp. 181, for any positive $\in$ we can find a constant M such that,

$$s^{m+1} \int_T^\infty e^{-st} \left| \alpha(t) - \frac{B t^m}{\Gamma m+1} \right| dt < \frac{M\, s^{m+1}}{(s-\in) e^{(s-\in)T}}, \quad for\, s > \in$$

Where the right hand side of this inequality approaches to zero as s becomes infinite. Therefore,

$$\lim_{s\to\infty} \left| s^{m+1} LW\{f(t,\Im)\} - B \right| \le \lim_{s\to\infty} s^{m+1} \left\{ \int_0^T \int_0^\tau \left| f(t,\Im) - \frac{B t^m \Im^n}{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}} dt\, d\Im \right\}$$

$$\le \sup_{\substack{0\le t\le T \\ 0\le \Im \le \tau}} \left| \frac{(2)^n \Gamma m+1 \Gamma n+\frac{1}{2}}{t^m \Im^n} f(t,\Im) - B \right|$$

Since T and τ are arbitrary,

$$\lim_{s\to\infty}\left|s^{m+1}\,LW\{f(t,\Im)\}-B\right| \le \lim_{\substack{t\to 0^+\\ \Im\to 0^+}}\left|\frac{(2)^n\,\Gamma m+1\,\Gamma n+\frac{1}{2}}{t^m\,\Im^n}\,f(t,\Im)-B\right|$$

From which the result follows.

2.2 Lemma

If $f(t,y)\in LW_{a,b,\alpha}$ with its support in $t_f \le t<\infty$ and $y_f \le y<\infty$ where $t_f >0$ and $y_f >0$ then $|F(s,x)|\le H\,e^{at-\frac{by}{2}+\frac{y^2}{4}}$, where H is sufficiently large constant.

Proof:- Let $g(t,y)$ be a smooth function on $0\le t\le\infty$ and $0\le y\le\infty$ such that $g(t,y)=1$ on $\lfloor t_f,\infty)$ and $\lfloor y_f,\infty)$ and also $g(t,y)=0$ on $(0,T)$ and $(0,Y)$ where $T<t_f$ and $Y<y_f$. As a distribution of slow growth satisfies a boundedness property of distribution, there exists a positive constant K and a non-negative integer η such that,

$$|F(s,x)|\le K\max_{0\le t\le\eta}\ \sup_{\substack{0\le t,\\ y<\infty}}\left|e^{at-\frac{by}{2}+\frac{y^2}{4}}\,D_t^p\,D_y^q\,g(t,y)\,f(t,y)\right|$$

$$\le K\max_{0\le t\le\eta}\ \sup_{\substack{0\le t,\\ y<\infty}}\left|e^{at-\frac{by}{2}+\frac{y^2}{4}}\,g^{p+q}(t,y)\,f(t,y)\right|$$

Where $g^{p+q}(t,y)$ gives p^{th} derivative of $g(t,y)$ with respect to 't' and q^{th} derivative of $g(t,y)$ with respect to 'y'

$$|F(s,x)|\le K\max_{0\le t\le\eta}\ \sup_{\substack{0\le t,\\ y<\infty}} e^{at-\frac{by}{2}+\frac{y^2}{4}}$$

$$\le H\,e^{at-\frac{by}{2}+\frac{y^2}{4}}$$, where H is sufficiently large constant.

3. Final value theorem for Laplace-Weierstrass transform

For a locally integrable function $f(t,\Im)$ satisfying above conditions with $T=0$ and existence of any complex constant B and a real numbers m and n such that,

$i)\ m>-1$

$ii)\ n>-1$

$$iii)\ \lim_{\substack{t\to\infty\\ \Im\to\infty}}\frac{\Gamma m+1\ \Gamma n+\frac{1}{2}\,(2)^n}{t^m\,\Im^n}\,f(t,\Im)=B$$

then, $\lim_{s\to 0^+} s^{m+1}LW\{f(t,\Im)\}=B$

Proof: We proceed as in initial value theorem for Laplace-Weierstrass transform to obtain,

$$\left|s^{m+1}LW\{f(t,\Im)\}-B\right| \le s^{m+1} \{\int_0^T\int_0^\tau \left| f(t,\Im)-\frac{Bt^m\Im^n}{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}}dt\,d\Im$$

$$+\int_T^\infty\int_0^\tau \left| f(t,\Im)-\frac{Bt^m\Im^n}{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}}dt\,d\Im$$

$$+\int_0^T\int_\tau^\infty \left| f(t,\Im)-\frac{Bt^m\Im^n}{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}}dt\,d\Im$$

$$+\int_T^\infty\int_\tau^\infty \left| f(t,\Im)-\frac{Bt^m\Im^n}{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}} \right| e^{-st-\frac{\Im^2}{4}}dt\,d\Im$$

Therefore,

$$\lim_{s\to 0^+}\left|s^{m+1}LW\{f(t,\Im)\}-B\right| \le \lim_{s\to 0^+} s^{m+1}\left\{\int_T^\infty\int_\tau^\infty \frac{t^m\Im^n}{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}}\left|\frac{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}\,f(t,\Im)}{t^m\Im^n}-B\right| e^{-st-\frac{\Im^2}{4}}dt\,d\Im\right\}$$

$$\le \underset{\substack{T\le t\le\infty \\ \tau\le\Im\le\infty}}{Sup}\left|\frac{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}}{t^m\,\Im^n}f(t,\Im)-B\right|$$

Since T and τ are arbitrary,

$$\lim_{s\to 0^+}\left|s^{m+1}LW\{f(t,\Im)\}-B\right| \le \lim_{\substack{t\to\infty \\ \Im\to\infty}}\left|\frac{(2)^n\Gamma m+1\,\Gamma n+\frac{1}{2}}{t^m\,\Im^n}f(t,\Im)-B\right|$$

4. Conclusion

This paper provides Initial and Final value theorems which can be used to solve boundary value problem.

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